Modern Physics

-*Black-body radiation*-

Lecture notes by
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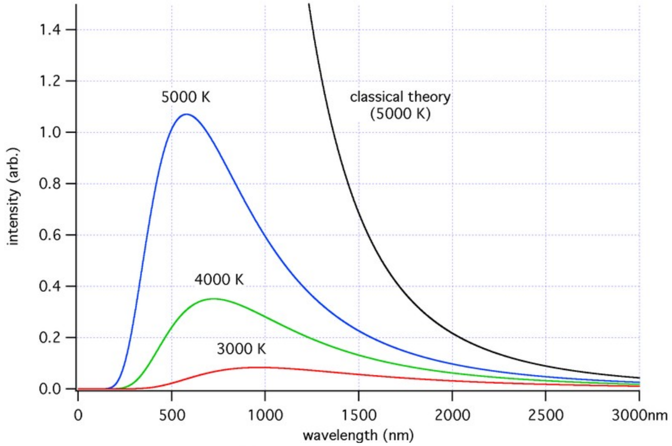
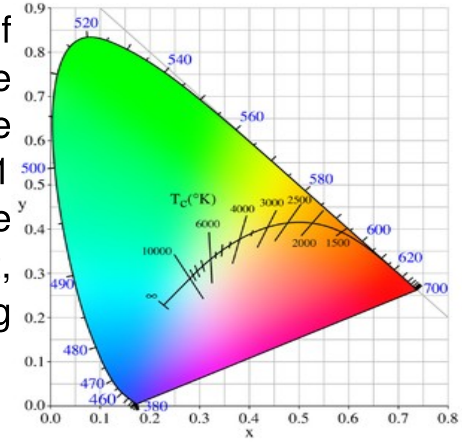
Origin of the Planck's constant: the black-body radiation

•A black body is an idealized object that absorbs all electromagnetic radiation that falls on it. No electromagnetic radiation passes through it and none is reflected. Because no light (visible electromagnetic radiation) is reflected or transmitted, the object appears black when it is cold. However, a black body emits a temperature-dependent spectrum of light. This thermal radiation from a black body is termed black-body radiation.

At room temperature, black bodies emit mostly infrared light, but as the temperature increases past a few hundred degrees Celsius, black bodies start to emit visible wavelengths, from red, through orange, yellow, and white before ending up at blue, beyond which the emission includes increasing amounts of ultraviolet. The term "black body" was introduced by Gustav Kirchhoff in 1860. Black-body emission gives insight into the thermal equilibrium state of a continuous field. In classical physics, each different Fourier mode in thermal equilibrium should have the same energy, leading to the theory of ultraviolet catastrophe that there would be an infinite amount of energy in any continuous field



Perception of color of black body by the human eye using the conventional CIE 1931 color space (our cone receptors feel short, middle and long wavelength)



Radiation inside a cavity – Rayleigh and Jeans

A black-body is seen as a cavity of a typical linear dimension L with perfectly conducting walls. The inner oscillating electromagnetic field must satisfy the boundary conditions (zero field at the walls). The oscillating electric fields will have nodes at the border of the cavity.

- Here ν_x,ν_y,ν_z is a set of positive integers.

$$E_x^{\nu_x\nu_y\nu_z}(\vec{r},t) = E_x^0(k_{\nu_x\nu_y\nu_z})\cos(\pi\nu_x x/L)\sin(\pi\nu_y y/L)\sin(\pi\nu_z z/L)\exp(i\omega_{\nu_x\nu_y\nu_z}t)$$
$$E_y^{\nu_x\nu_y\nu_z}(\vec{r},t) = E_y^0(k_{\nu_x\nu_y\nu_z})\sin(\pi\nu_x x/L)\cos(\pi\nu_y y/L)\sin(\pi\nu_z z/L)\exp(i\omega_{\nu_x\nu_y\nu_z}t)$$
$$E_z^{\nu_x\nu_y\nu_z}(\vec{r},t) = E_z^0(k_{\nu_x\nu_y\nu_z})\sin(\pi\nu_x x/L)\sin(\pi\nu_y y/L)\cos(\pi\nu_z z/L)\exp(i\omega_{\nu_x\nu_y\nu_z}t)$$

- To satisfy the Helmoltz equation

(wave equation in space)

$$k^2_{\nu_x\nu_y\nu_z} = (\pi\nu_x/L)^2 + (\pi\nu_y/L)^2 + (\pi\nu_z/L)^2 = (\omega_{\nu_x\nu_y\nu_z}/c)^2$$

The cycle-averaged value of the energy density associated with a particular mode is:

$$W(k_{\nu_x\nu_y\nu_z}) = \frac{1}{4V}\int_{\text{cavity}}\left[\varepsilon_0\left|\vec{E}^{\nu_x\nu_y\nu_z}(\vec{r},t)\right|^2 + \mu_0\left|\vec{H}^{\nu_x\nu_y\nu_z}(\vec{r},t)\right|^2\right]dV$$
$$= \frac{1}{2V}\int_{\text{cavity}}\left[\varepsilon_0\left|\vec{E}^{\nu_x\nu_y\nu_z}(\vec{r},t)\right|^2\right]dV = \frac{1}{16}\varepsilon_0\left|\vec{E}^0(k_{\nu_x\nu_y\nu_z})\right|^2$$

For a thermal source, the most significant experimentally measurable object is the frequency distribution of the stored energy density. To obtain this distribution, we take the energy density:

$$W(\omega)\,d\omega = \sum_{\{\nu_x,\nu_y,\nu_z\}} W(k_{\nu_x\nu_y\nu_z}) \Leftarrow \left\{\text{with } k_{\nu_x\nu_y\nu_z} \text{ between } \omega/c \text{ and } (\omega+d\omega)/c\right\}$$
$$= \left\langle W(k_{\nu_x\nu_y\nu_z})\right\rangle \times \begin{bmatrix}\text{Number of modes with frequencies}\\ \text{between } \omega \text{ and } \omega+d\omega\end{bmatrix}$$

Number of modes

We need to know the density of modes with frequencies in the interval $\Delta\omega$. Let's suppose L >> λ of the waves, so there will be many, billions of modes.
Each mode is a sinusoidal wave going to zero at the ends, so there will be an integral number of half-wavelengths in the length L. We use the wave vector k= $2\pi/\lambda$ so we have a discretization of the wave vectors: kj = $j\pi/L$.
The separation δk is obviously π/L. If L >> λ there will be many modes in an interval Δk, and their number is given by:

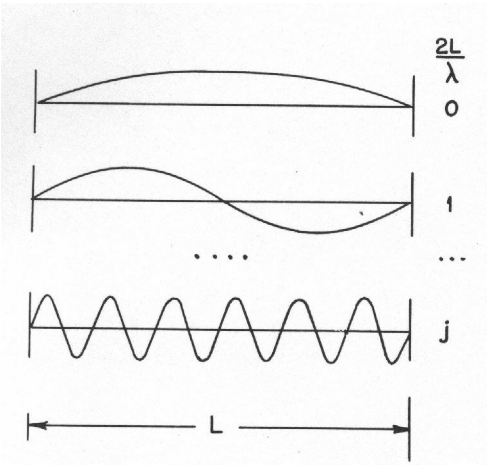


Fig. 4–8. The standing wave modes on a line.

$$\Delta N(k) = \left(\frac{\Delta k}{\delta k}\right) = \left(\frac{L}{\pi}\right) \Delta k$$

In reality, this counts standing waves contains two modes k and -k (in opposite directions). We usually prefer to count states for each k so we have to take a number half as big:

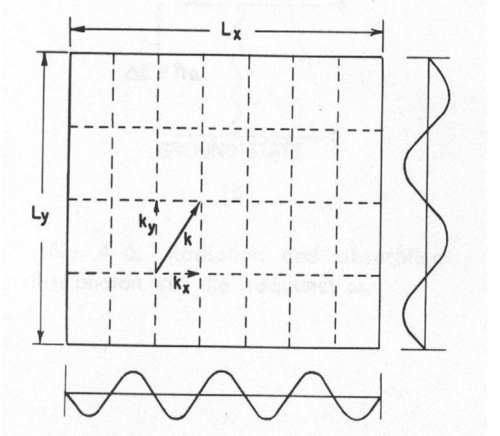


Fig. 4–9. Standing wave modes in two dimensions.

$$\Delta N(k) = \left(\frac{L}{2\pi}\right) \Delta k$$

$$\Delta N(\mathbf{k}) = \left(\frac{L_x L_y L_z}{(2\pi)^3}\right) (\Delta k_x \Delta k_y \Delta k_z)$$

Number of light modes

In 3 dimensions we have then that the number of modes in a cavity is proportional to the volume V and to the volume in wave-vector space, a general result. In the case of electromagnetic waves, light modes, we have also to account for the two different polarizations (on the plane orthogonal to the propagation direction) so the number of states is multiplied by two.

The simple relationship between ω and k ($k = \omega/c$) is used to write the number of states included in an interval dω :

$$dN(\mathbf{k}) = 2 \cdot \left(\frac{V}{(2\pi)^3}\right) d^3\mathbf{k}$$

$$dN(\omega) = 2 \cdot \left(\frac{V}{(2\pi)^3}\right) 4\pi k^2 dk$$

$$dN(\omega) = \left(\frac{V}{(2c\pi)^3}\right) 8\pi\omega^2 d\omega = \left(\frac{V\omega^2}{(\pi)^2 c^3}\right) d\omega$$

The ultra-violet catastrophe

Following the traditional approach of using the "theorem of equipartion of the energy" used by Rayleigh and Jeans we can write the average energy density of the oscillation mode to be described by its thermodynamical average: <W> = $K_B T/V$ (2 degrees of freedom), in this way we arrive to the Rayleigh-Jeans radiation law:

$$< W_T(\omega) > d\omega = K_b T \left(\frac{\omega^2}{(\pi)^2 c^3} \right) d\omega$$

$$< W_T(\omega) >= K_b T \left(\frac{8\pi\nu^2}{c^3} \right)$$

$$< W_T(\lambda) >=< W_T(\omega) > \left(\frac{d\nu}{d\lambda} \right) =$$

$$=< W_T(\omega) > \left(\frac{d(c/\lambda)}{d\lambda} \right) =< W_T(\omega) > \left(\frac{-c}{\lambda^2} \right)$$

$$< W_T(\lambda) >= K_b T \left(8\pi\lambda^{-4} \right)$$

The energy density, corresponding to the radiated spectrum of the black-body,` increases with the square of the frequency (or with the inverse 4^{th} power of wavelength)!

Black-body emission gives insight into the thermal equilibrium state of a continuous field. In classical physics, each different Fourier mode in thermal equilibrium should have the same energy, leading to the theory of ultraviolet catastrophe that there would be an infinite amount of energy in any continuous field.

Planck's background

The aim of Plank at that time (end of the XIX century) was to explain some known features of the black-body spectrum:

1) Wien's displacement law (Nobel prize 1911 for heat radiation) for which there is a maximum emission at a wavelength inversely proportional to T:

$$\lambda_{max} = \frac{hc}{x}\frac{1}{kT} = \frac{2.89776829\ldots \times 10^6 \text{ nm} \cdot \text{K}}{T}$$

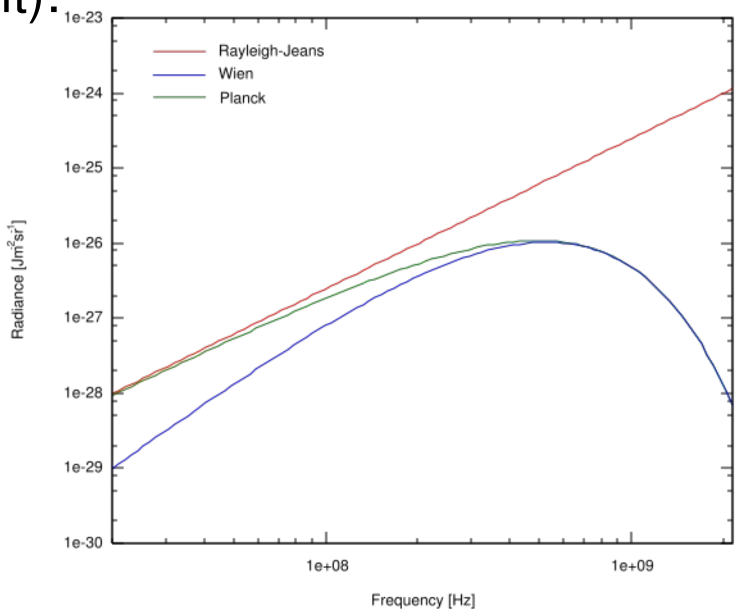
2) Wien's empirical law (later shown to be consistent with electromagnetism and thermodynamics by Planck) for the radiated intensity I (energy passing through a unit surface, in unit time, and for solid angle unit):

$$I(\nu, T) = < W_T(\omega) > \cdot \left(\frac{A \cdot c \cdot t}{A \cdot t\Omega}\right)$$

$$I(\nu, T) = < W_T(\omega) > \cdot \left(\frac{c}{4\pi}\right)$$

$$I(\nu, T) = h\nu e^{-\beta h\nu}\left(\frac{8\pi\nu^2}{c^3}\right) \cdot \left(\frac{c}{4\pi}\right)$$

$$I(\nu, T) = he^{-\beta h\nu}\left(\frac{2\nu^3}{c^2}\right)$$



Planck's hypotesis

- The basic Planck's assumption (Annalen Der Physik, 4, 553 1901, Nobel Prize 1918) was that the energy of the black-body radiation field could be exchanged only through well-defined energy quanta, proportional to the frequency of the associated electromagnetic wave **E= h ν** .This assumption was enough to reconcile the calculations with the experimental data and the consequences of this quantization on classical physics were not realized by Planck himself.
- *The basic idea was to calculate the energy of an ensemble of "oscillators" or "resonators" that could exchange only energies* ***h ν*** *in a thermal bath.*
- The derivation made by Planck can be today summarized in this way.
- The probability P_n that the mode (oscillator) is thermally excited to an energy E_n is:

$$P_n = \frac{\exp(-\beta E_n)}{\sum_n \exp(-\beta E_n)} = \frac{\exp(-\beta n \hbar \omega)}{\sum_n \exp(-\beta n \hbar \omega)} = \exp(-\beta n \hbar \omega)\left[1 - \exp(-\beta \hbar \omega)\right]$$

Planck's derivation

•The mean value <n>, number of excited oscillation modes, in the thermal bath T can be calculated in an analgous way

•It is found to depend only on the hν/KT ratio

•It is found in general different from the classical distribution of Maxwell-Boltzmann exp(-βE), valid for the distribution (continuous) of the energy of a rarefied gas (and in any classical system where we know the Energy function)

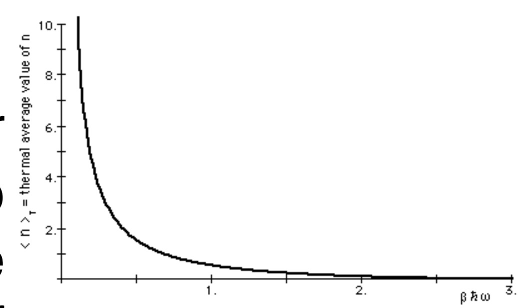
•It tends to the "classical" results for large E/KT values larger than 1 (so Energy exchange and Temperature should be at least of the same order of magnitude).

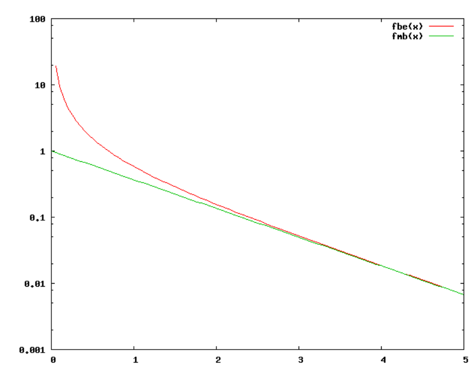
$$\langle n \rangle_T = \sum_s n P_s = \frac{\sum_s n \exp(-\beta n \hbar\omega)}{\sum_s \exp(-\beta n \hbar\omega)}$$

$$= -\{\partial/\partial(\beta \hbar\omega)\} \ln\left[\sum_s \exp(-\beta n \hbar\omega)\right]$$

$$= -\{\partial/\partial(\beta \hbar\omega)\} \ln\left[\{1 - \exp(-\beta \hbar\omega)\}^{-1}\right]$$

$$= \frac{1}{\exp(\beta \hbar\omega) - 1}$$

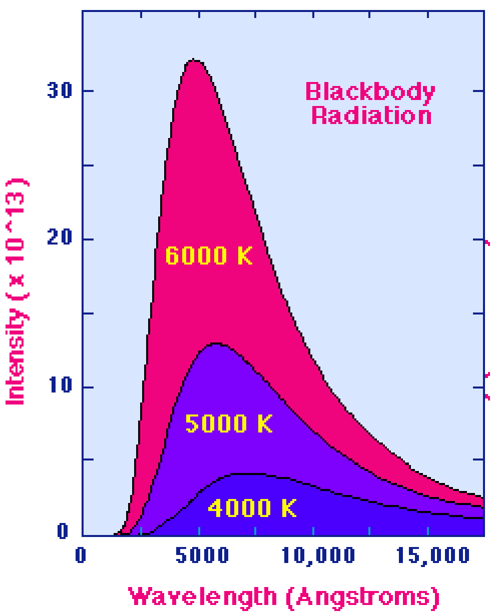




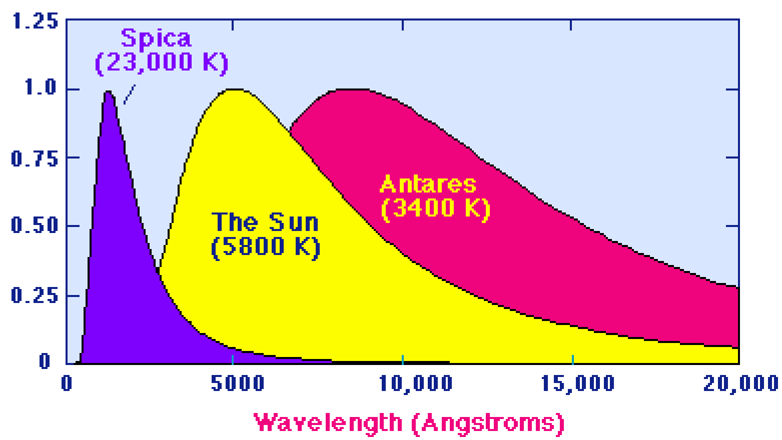
Plancks' radiation law

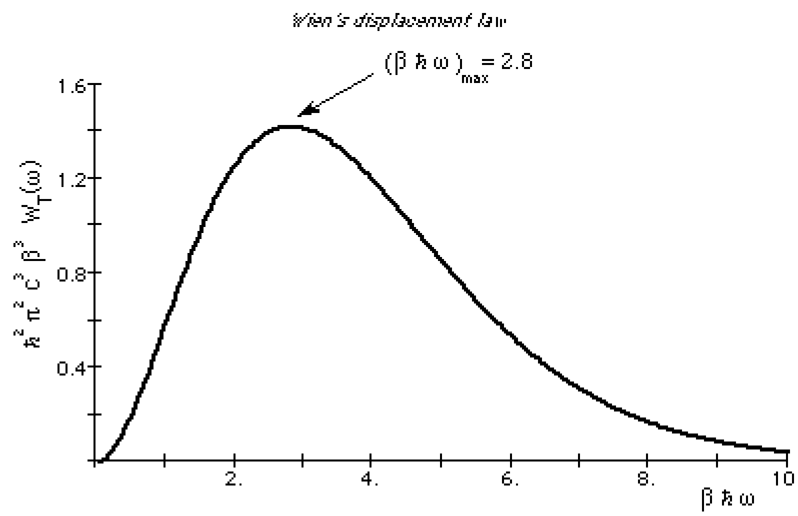
- The energy spectrum of the blackbody is simply found by multiplying the quantum energy by the average number of excited modes and by the number of different modes contained in an interval δω.

$$< W_T(\nu) >= h\nu \left[\frac{1}{e^{\beta h\nu} - 1}\right] \left(\frac{8\pi\nu^2}{c^3}\right) \qquad u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}.$$



$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}.$$





Stefan- Boltzmann law

- Stefan derived an expression for the integrated power per unit surface emitted by a black body from available experimental data in 1879. The law was derived by Boltzmann by using thermodynamical arguments.
- The constant σ was found to be the same for any system approximating a black-body. For a “grey body” a total emissivity factor ε (<1) can be defined.
- The S-B law can be derived by integrating the energy density calculated by Planck.

$$j^{\star} = \sigma T^4.$$

$$j^{\star} = \epsilon \sigma T^4.$$

$$j^{\star} = \int_0^{\infty} \left(\frac{dj^{\star}}{d\lambda} \right) d\lambda$$

$$u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}. \quad I(\nu, T) = \frac{2h\nu^3}{c^2} \; \frac{1}{e^{h\nu/kT} - 1}.$$

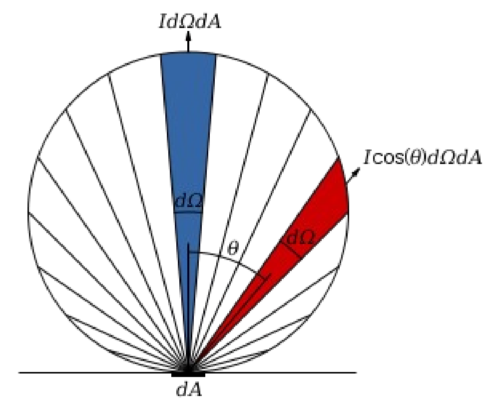
- The power radiated at frequency ν by a surface A on a solid angle Ω for a temperature T is: $I(\nu, T)\ A\ d\nu\ d\Omega$
- The Stefan law is related to this integral, where the integration is extended to half the solid angle:

$$\frac{P}{A} = \int_0^{\infty} I(\nu, T) d\nu \int d\Omega$$

Planck's derivation of S-B law

To derive the Stefan–Boltzmann law, we must integrate Ω over the half-sphere and integrate ν from 0 to ∞. Furthermore, the intensity observed along the sphere will be the actual intensity times the cosine of the zenith angle φ (Lambert's cosine law), and in spherical coordinates, dΩ = sin(φ) dφ dθ

$$\frac{P}{A} = \int_0^\infty I(\nu,T)d\nu \int d\Omega = \int_0^\infty I(\nu,T)d\nu \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\phi \sin\phi d\phi = \pi \int_0^\infty I(\nu,T)d\nu$$

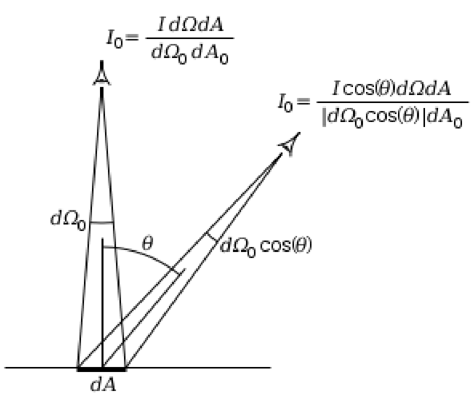


$$\frac{P}{A} = \frac{2\pi h}{c^2}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{kT}}-1}d\nu \qquad u = \frac{h\nu}{kT} \qquad \frac{P}{A} = \frac{2\pi h}{c^2}\left(\frac{kT}{h}\right)^4 \int_0^\infty \frac{u^3}{e^u-1}du.$$

$$J = \int_0^\infty \frac{x^3}{\exp(x)-1}\,dx = \frac{\pi^4}{15}.$$

$$j^\star = \frac{2\pi^5 k^4}{15h^3c^2}T^4 \qquad j^\star = \sigma T^4.$$

$$\sigma = \frac{2\pi^5 k^4}{15c^2h^3} = 5.670400 \times 10^{-8}\,\mathrm{J\,s^{-1}m^{-2}K^{-4}}$$

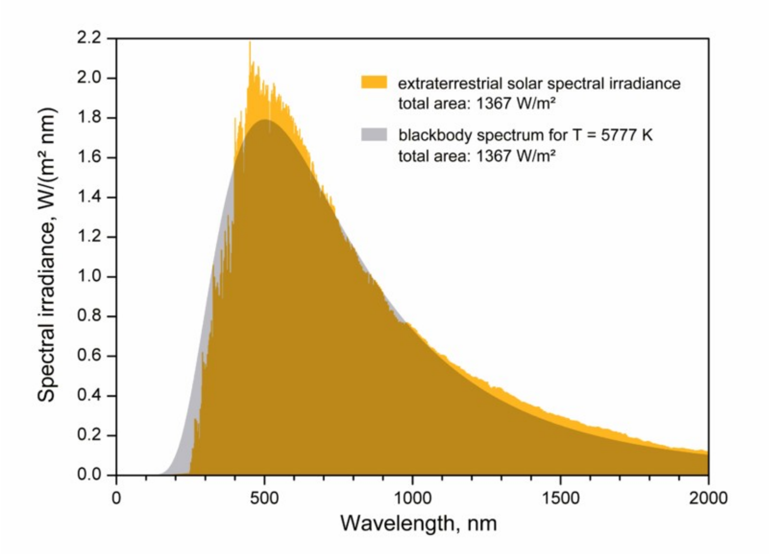


Thus at 100 K the energy flux density is 5.67 W/m2, at 1000 K 56.700 W/m2, etc.

Important applications

Temperature of the Sun

With his law Stefan also determined the temperature of the Sun's surface. He learned from the data of Charles Soret (1854–1904) that the energy flux density from the Sun is 29 times greater than the energy flux density of a warmed metal sheet. A round sheet was placed at such a distance from the measuring device that it would be seen at the same angle as the Sun. Soret estimated the temperature of the lamella to be approximately 1900 °C to 2000 °C (1950 °C with a 50 °C uncertainty). Stefan surmised that ⅓ of the energy flux from the Sun is absorbed by the Earth's atmosphere, so he took for the correct Sun's energy flux a value 3/2 times greater, namely 29 × 3/2 = 43.5. Precise measurements of atmospheric absorption were not made until 1888 and 1904.



The temperature used for the sheet was the absolute thermodynamic one (1950+273 K). As $2.57^4 = 43.5$, it follows from the law that the temperature of the Sun is 2.57 times greater than the temperature of the sheet, so Stefan got a value of 5430 °C or 5700 K (modern value is 5780 K). This was the first sensible value for the temperature of the Sun. Before this, values ranging from as low as 1800 °C to as high as 13,000,000 °C were claimed.

Temperature of (distant) stars

Temperature of stars

The temperature of stars other than the Sun can be approximated using a similar means by treating the emitted energy as a black body radiation.

So:

$$L = 4\pi R^2 \sigma T_e^4$$

where L is the luminosity, taking into account the radiating surface, σ is the Stefan-Boltzmann constant, R is the stellar radius and T is the effective temperature. T

The surface temperature of a star can be nowadays better determined by observing its spectrum and identifying the maximum (and the shape) of the distribution.

Knowing the temperature, the S-B equation can be also used to compute the approximate radius of a main sequence star relative to the sun, just measuring the luminosity L of the star:

$$\frac{R}{R_\odot} \approx \left(\frac{T_\odot}{T}\right)^2 \cdot \sqrt{\frac{L}{L_\odot}}$$

where R_o is the solar radius. With the Stefan–Boltzmann law, astronomers can easily infer the radii of stars.

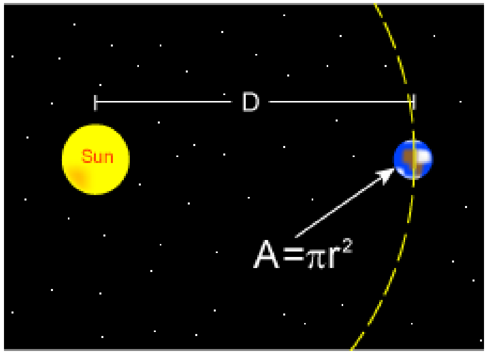
Temperature of the earth

Temperature of the Earth

We can calculate the effective temperature of the Earth T by equating the energy received from the Sun and the energy transmitted by the Earth, under the black-body approximation. The total power emitted by the Sun can be approximated as:

$$P_{Semt} = \left(\sigma T_S^4\right)\left(4\pi R_S^2\right) \qquad (1)$$

The earth is hitten only by a small fraction of the radiated energy, which is partly reflected to space (albedo α):



$$P_{Eabs} = P_{Semt}(1-\alpha)\left(\frac{\pi R_E^2}{4\pi D^2}\right) \qquad (2)$$

On the other hand the earth is also emitting radiation:

$$P_{Eemt} = \left(\sigma T_E^4\right)\left(4\pi R_E^2\right) \qquad (3)$$

We consider the earth in thermal equilibrium so:

$$\left(\sigma T_S^4\right)\left(4\pi R_S^2\right)(1-\alpha)\left(\frac{\pi R_E^2}{4\pi D^2}\right) = \left(\sigma T_E^4\right)\left(4\pi R_E^2\right).$$

Therefore the estimated average temperature of the earth under present condition is:

$$T_E = T_S\sqrt{\frac{\sqrt{1-\alpha}R_S}{2D}}$$

Temperature of the earth

Temperature of the Earth

Using the known values for the Sun

$$T_E = T_S \sqrt{\frac{\sqrt{1-\alpha} R_S}{2D}}$$

Ts=5778 K

Rs=6.96 10^{8} m (109 R_E)

D= 1.496 10^{11} m

We obtain a reasonable estimate for the temperature: T_E = 249 K (- 24 C), using an average "albedo" α = 0.36.

The actual average surface temperature below the gaseous atmosphere is higher (~ 287 K) due to the well-known greenhouse effect.

Estimates of the Earth's average albedo vary in the range 0.3–0.4, resulting in different estimated effective temperatures. Estimates are often based on the solar constant (total insolation power density) rather than the temperature, size, and distance of the sun. For example, using 0.4 for albedo, and an insolation of 1400 Wm-2), one obtains an effective temperature of about 245 K.