Modern Physics
-quanta of light-

Lecture notes by
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Background: photoelectric effect and blackbody radiation

•The revolution imposed by the discovery that photoeletric effect and black-body radiation could be explained only assuming quantized exchange of energy with the electromagnetic field prompted many scientists to discuss and put to a test this new idea: a quantized radiation field.

•Many controversies appeared after 1905 on the interpretation of this quantization but a series of theoretical papers (by Einstein and others) and experimental studies showed some crucial aspects of the new quantum theory of radiation and among them:

1) Extension of the “photon” concept (the name *photon* was finally given by Lewis in 1926 – although to a different entity) including a well-defined momentum

$p=h\nu/c = h/\lambda$ (Einstein 1916, experiment by Compton with x-rays 1923).

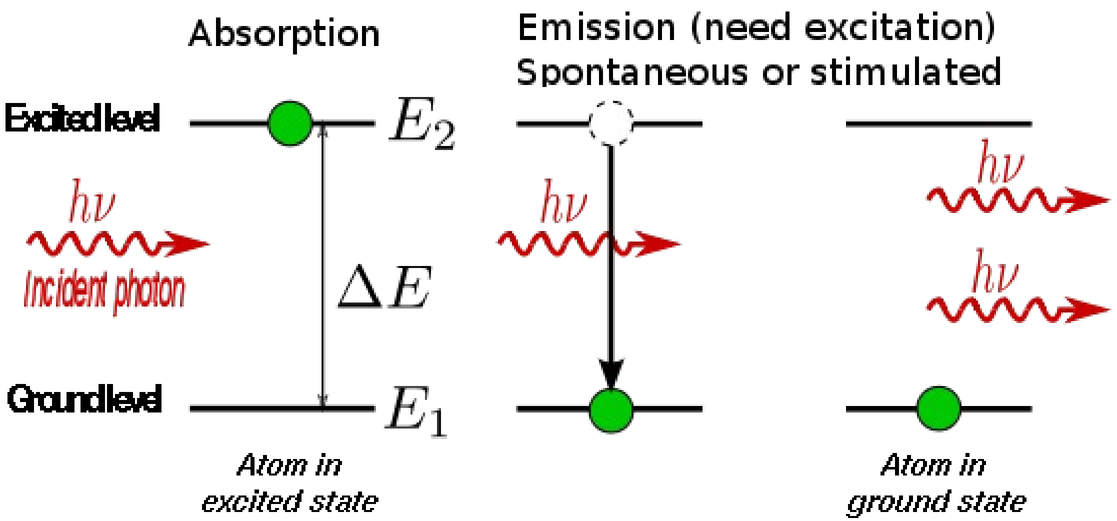
2) Derivation of the Planck's black-body radiation formula assuming a direct energy exchange between the e.m. Field (photons) and the atoms of the cavity (absorption, spontaneous and stimulated emission of radiation, Einstein 1917).

Einstein's derivation of the Planck's blackbody formula

This demonstration is based on a combination of concepts from thermodynamics, statistics and electromagnetism. The basic assumption is that light is composed by particles, and that the atoms are systems of charged particles (electrons and nuclei) with discrete energy levels. This last assumption was justified by many experimental evidence concerning the atomic structure (Rydberg 1888, Rutherford 1911, Bohr 1913, Sommerfeld 1915).

- The basic processes for the photon-atom interaction are shown in the figure: absorption of one photon; spontaneous emission of a photon within the lifetime of the atom excited state; stimulated (induced) emission of photons (needs an excited state).

- The energy exchange mechanism between the radiation field (photon emission and absorption) and the atoms of the cavity in thermal equilibrium is the basis for the calculation of the black-body radiation.



$$E_2 - E_1 = \Delta E = h\nu$$

Transition rates

Let us consider a system of N atoms, for which N_1 atoms are in the level E_1 and N_2 are in level E_2. This system is considered in thermal equilibrium (no exchange of energy or particles is present with any external system). Interaction with the radiation field present within such a system is possible only as an emission or absorption of discrete energy quanta $h\nu = E_2 - E_1$.

The radiation field is taken to have the energy density u(ν,T), u(ν) for a given temperature T. According to the theory, the following transitions are produced per unit time:

•**Absorption 1->2**: The number of processes in time dt is proportional to the occupation number N_1 of level 1 and to the energy density of the e.m. Field *u(ν)*:

$$dN_{12} = B_{12}\, u(\nu)\, N_1\, dt$$

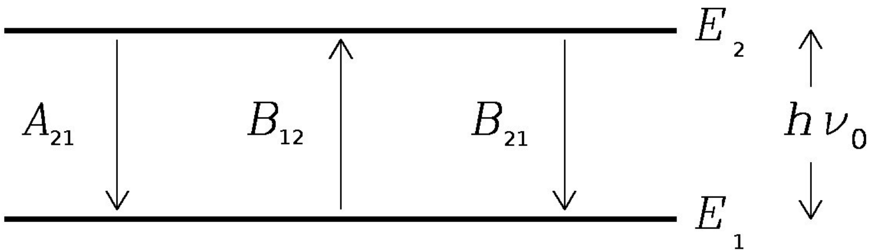
•**Emission (spontaneous) 2->1**: the number of this emission processes (not related to the radiation) per unit time must be proportional to the occupation number:

$$dN'_{21} = A_{21}\, N_2\, dt$$

•**Emission (stimulated) 2->1**: the number of emission processes induced by the existence of the radiation field is obviously proportional to the number and to the energy density *u*:

$$dN''_{21} = B_{21}\, u(\nu)\, N_2\, dt$$

The A, B coefficients are called Einstein coefficients and measure the transition probabilities.



Quantum States and Thermal equilibrium

Under thermal equilibrium conditions an equal number of transitions 1->2 and 2->1 must occur: $dN_{12} = dN'_{21} + dN''_{21}$ (the population of the states is constant)

Using the expressions for the variations dN we obtain:

$$dN_{12} = B_{12}\, u(\nu)\, N_1\, dt = dN'_{21} + dN''_{21} = A_{21}\, N_2\, dt + B_{21}\, u(\nu)\, N_2\, dt$$

$$B_{12}\, u(\nu)\, N_1 = A_{21}\, N_2 + B_{21}\, u(\nu)\, N_2 \qquad \frac{N_2}{N_1} = \frac{B_{12}\, u(\nu)}{A_{21} + B_{21}\, u(\nu)}$$

Since the system is in thermal equilibrium, the probability to have an energy En is proportional to the Boltzmann factor exp (-En/KT), the ratio of occupation numbers must be proportional to the ratio of the Boltzmann factors:

$$\frac{N_2}{N_1} = \frac{e^{-E_2/KT}}{e^{-E_1/KT}} = \frac{B_{12}\, u(\nu)}{A_{21} + B_{21}\, u(\nu)} \qquad u(\nu) = \frac{A_{21}}{B_{12}\, e^{h\nu/KT} - B_{21}}$$

So we find the relationship between the energy density of the radiation field and Einstein coefficients A, B (where we have written $h\nu = E_2 - E_1$).

Classical limits for the energy density

The A, B coefficients can be determined by considering the energy density behaviour for infinite temperature and for small frequencies hν << KT for which the classical Rayleigh-Jeans law must hold.

1. The energy density is unlimitedly large for unlimited large temperatures:

$$\lim_{T\to\infty} u(\nu,T) = \lim_{T\to\infty} \frac{A_{21}}{B_{12}\, e^{h\nu/KT} - B_{21}} = \frac{A_{21}}{B_{12} - B_{21}} \to \infty \to B_{12} = B_{21}$$

2. The Rayleigh-Jeans tell us that:

$$u(\nu,T) = \frac{8\pi\nu^2 K_B T}{c^3} = \frac{A_{21}}{B_{12}(e^{h\nu/KT} - 1)}$$

On the other hand we have

$$e^{h\nu/KT} = 1 + \frac{h\nu}{K_B T} + \ldots \qquad \text{so} \qquad u(\nu,T) = \frac{A_{21}}{B_{12}} \frac{K_B T}{h\nu} \qquad h\nu \ll K_B T$$

Therefore, we have in general that: $\frac{A_{21}}{B_{12}} = \frac{8\pi h \nu^3}{c^3}$ which states that the probabilities of spontaneous emission and absorption are proportional throught the 3rd power of the frequency. Absorption and stimulated emission are complementary processes ($B_{12}=B_{21}$).

Planck's results using photons

Using the previous results we can easily derive the expression of the energy density of the radiation field:

$$u(\nu)=\frac{A_{21}}{B_{12}\,e^{h\nu/KT}-B_{21}}$$

$$\frac{A_{21}}{B_{12}}=\frac{8\pi h\nu^3}{c^3} \qquad B_{21}=B_{12}$$

We obtain the Planck's law:

$$u(\nu)=\frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/KT}-1}$$

This derivation shows in a particularly elegant way that the correct energy density for the black-body radiation field can be obtained by assuming the existence of light quanta (photons) interacting (exchanging energy) with the atoms of the cavity. The wave nature of the light field here is completely un-necessary!

The concept of stimulated emission is also at the basis of the theory of masers and lasers (cavity with reflecting mirrors), nowadays devices of common use.

Photon kinematic

- Light quanta behaving like particles are regarded by Einstein as having also a well-defined momentum p.
- The problem is that light quanta travel at the velocity of light and have no mass, while definition of the kinematics of the interaction of those particles with other particles (like electrons) require a fully relativistic treatment.
- The definition of energy and momentum of relativistic particles needs a proper development of the relativistic kinematic.
- Having defined a photon momentum and energy, we can describe elastic and inelastic interactions with other massive particles by using well-known conservation laws (energy, momentum).
- This provides the theoretical background for the Compton effect (inelastic scattering of photons by weakly bounded electrons in matter), observed at high photon energies (x-ray).

Summary of relativistic kinematic

Starting point: Einstein's postulates (1905)

•*1) The laws of physics are the same for all inertial reference frames (coordinate system moving at steady velocity v)*

•*2) The speed of light in vacuum is a constant (c) that does not depend upon the state of motion of the emitting body.*

To satisfy those postulates we find that the usual Galilean transformations among different inertial reference frames must be modified into the Lorentz transformations:

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x = \gamma\left(x' + vt'\right)$$

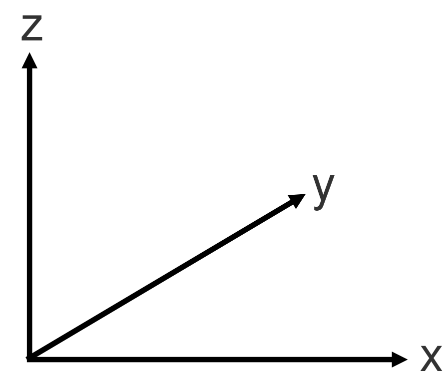
$$y = y'$$

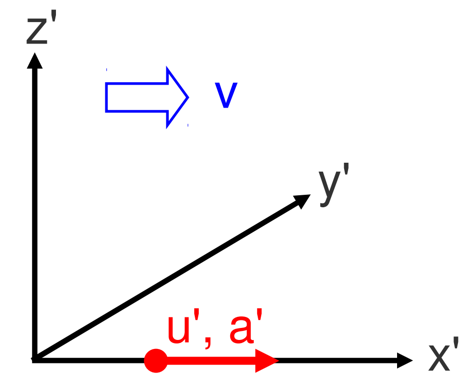
$$z = z'$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$\beta = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$





Lengths, time, velocity under Lorentz transformations

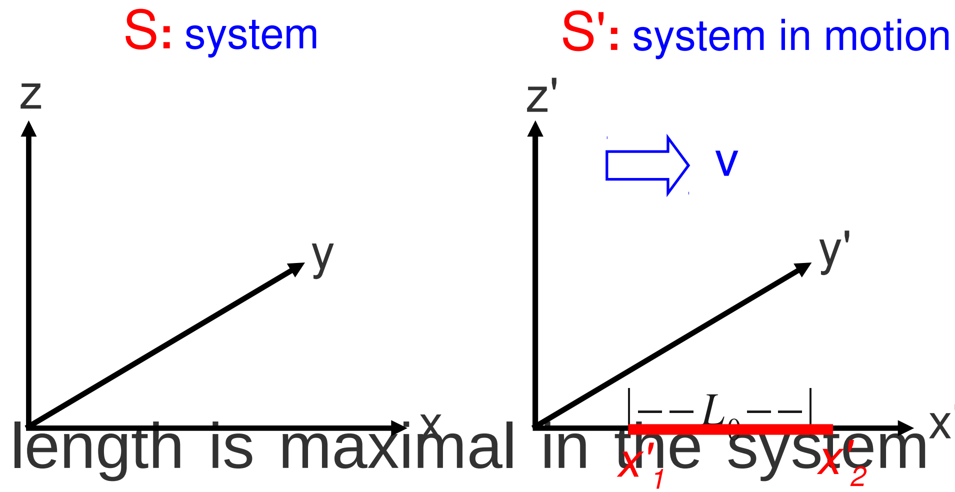
Using Lorentz transformations it can be easily shown that

$$L_0 = x'_2 - x'_1 \qquad L = x_2 - x_1$$

$$L_0 = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) =$$

$$L_0 = \gamma(x_2 - x_1) = \gamma L \qquad t_1 = t_2$$

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{v^2}{c^2}}$$



1) Lorentz length contraction: the length is maximal in the system where the rod (observer) is not moving.

$$x'_1 = x'_2$$

$$T_0 = t'_2 - t'_1 \qquad T = t_2 - t_1 = \gamma\left(t'_2 + \frac{v}{c^2}x'_2\right) - \gamma\left(t'_1 + \frac{v}{c^2}x'_1\right) = \gamma(t'_2 - t'_1) = \gamma T_0$$

2) Lorentz time dilatation: time intervals are minimal in the system where the rod (observer) is not moving.

$$T = \gamma T_0 = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Relativistic velocity transformations

Lorentz transformations

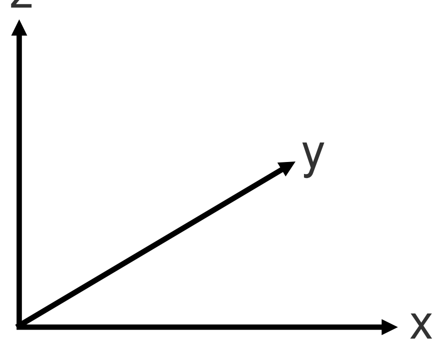
$$x' = \gamma (x - vt)$$

$$t' = \gamma \left(t - \frac{vx}{c^2} \right)$$

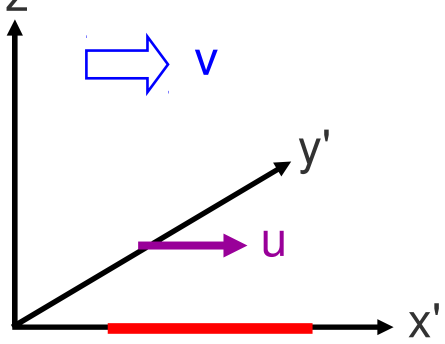
$u = \frac{dx}{dt}$ velocity of the particle in S

We search the velocity $u' = \frac{dx'}{dt'}$ Measured in S'

S: system



S': system in motion



$$\frac{dx'}{dt'} = \frac{\gamma (dx - \mathrm{v}\ dt)}{\gamma \left(dt - \frac{\mathrm{v}\ dx}{c^2} \right)} = \frac{\frac{dx}{dt} - v}{\left(1 - \frac{\mathrm{v}\ \frac{dx}{dt}}{c^2} \right)}$$

$$u' = \frac{u - \mathrm{v}}{1 - \frac{\mathrm{v}u}{c^2}}$$

And we find for the inverse transformation:

$$u = \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

Relativistic momentum conservation

Momentum conservation: this is a general principle that must be valid for any inertial reference system. But using Lorentz transformation p=mv is not a conserved quantity.

This prompt us to a re-definition of the momentum of a particle which satisfies the momentum conservation principle.

The relativistic momentum is defined as $p = m\frac{\Delta x}{\Delta t_0}$ where Δx is the interval seen by an external observer when the time interval in the reference system of the traveling object is Δt_0.

In this case we obtain: $$p = m\frac{\Delta x}{\Delta t_0} = m\frac{\Delta x}{\Delta t}\frac{\Delta t}{\Delta t_0} = m\frac{\Delta x}{\Delta t}\gamma$$

$$T = \gamma T_0$$

Time dilatation

Therefore the relativistic expression for the momentum becomes:

$$p = \gamma m v$$

Valid also for vectors: $$\vec{p} = \gamma m \vec{v}$$

This expression reduces to the classical one for v << c but increases rapidly for v ~c making possible to reach unlimitedly high value of p for velocities approaching that of light.

Mass-energy equivalence

If we consider the momentum expression we can recast the terms defining an effective relativistic mass which depends on the speed of the particle:

$p = m_0 \gamma v = m v \, ; \, m = m_0 \gamma \longrightarrow$ The relativistic mass increases and diverges approaching c! It is not possible to reach c for a massive particle.

Let us see the mass variation at moderate velocities: $\gamma = (1-\beta^2)^{(1/2)} \sim 1 + \frac{1}{2}\beta^2$

$$m = m_0 + \frac{1}{2} m_0 \frac{v^2}{c^2} \longrightarrow mc^2 = m_0 c^2 + \frac{1}{2} m_0 v^2 \longrightarrow \frac{1}{2} m_0 v^2 = mc^2 - m_0 c^2$$

$$E_{kin} = mc^2 - m_0 c^2 = (m_0 + \Delta m) c^2 - m_0 c^2 = \Delta m \, c^2$$

Therefore at moderate velocities we can define a total energy for the particle which is the sum of a rest energy m_0c^2 and a kinetic energy:

Mass is proportional to Energy!

$$E_{tot} = m c^2 = m_0 c^2 + \frac{1}{2} m_0 v^2$$

Energy-momentum relationship

The mass-energy relationship E = mc^2 derived above for moderate velocities can be shown to be true in general. This shows that a massive particles contains a well defined energy even at rest, and that the Energy becomes unlimited approaching the speed of light. A consequence is that a massive particle can not be accelerated to the velocity of light.

A relationship between momentum and energy can be also derived, and this is particularly useful for massless particles (like photons) having well-defined energies (and momenta).

$$p^2c^2 = m_0^2\gamma^2 v^2 c^2 = \frac{m_0^{\ 2}}{1-\beta^2}\beta^2 c^4 \longrightarrow p^2c^2 = -m_0^2\frac{(1-\beta^2)}{(1-\beta^2)}c^4 + \frac{m_0^2}{(1-\beta^2)}c^4$$

We obtain:

$$p^2c^2 = \frac{m_0^2}{(1-\beta^2)}c^4 - m_0^2c^4 = E_{tot}^2 - E_0^2$$

Therefore $E^2 = (m_0c^2)^2 + (pc)^2$

For massless particles this reduces to: $E = pc$

For photons (E= hν) we have :

$$E = pc = h\nu = h\frac{c}{\lambda} \qquad p = h\frac{\nu}{c} = \frac{h}{\lambda} = 2\pi\frac{\hbar}{\lambda} = \hbar k$$

X-ray experiments

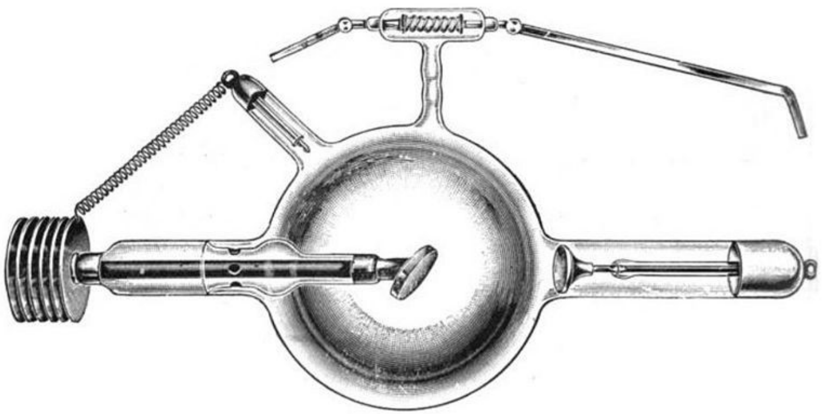
In 1923 A. H. Compton did an experiment providing the first direct proof of the photon momentum. The $p = h\frac{\nu}{c}$ relationship shows that it is important to increase the frequency (Energy) of the photon in order to measure differences in the momentum.

The x-rays were discovered by Roentgen in 1895 (first nobel prize in Physics, 1901) while working with various types of evacuated tubes in which electrical discharge were passing through them. The discovery was apparently accidental, and happened after that a Crookes-type tube subject to a discharge was covered by a light-tight cardboard screen (to prevent observing the light) was observed to induce a fluorescence on a screen put at some distance from the tube (about 1 m). Those rays were different in nature from the “cathod” rays described before (electrons) and were emitted in (almost) any direction with a very high penetration range.

The hands on the left were the first x-ray images collected by Roentgen on a photographic plate and made great impression.

X-ray generation: early times

X-rays are produced anytime high-energy electrons ("cathod" rays) hit matter (typically the anode). x-rays were discovered radiating from experimental discharge tubes called Crookes tubes.These were the first x-ray tubes used also for first medical applications. These first generation cold cathode or Crookes x-ray tubes were used until the 1920s. Crookes tubes generated the electrons needed to create x-rays by ionization of the residual air in the tube, instead of a heated filament.



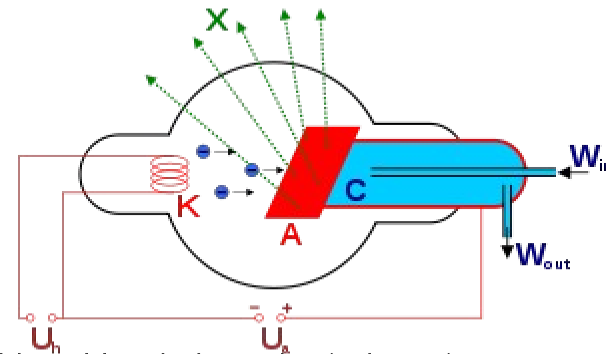
Crookes X-ray tube from early 1900s. The cathode is on the right, the anode is in the center with attached heat sink at left. The electrode at the 10 o'clock position is the 'auxiliary anode'. The device at top is a 'softener' used to regulate the gas pressure.

They consisted of a glass bulb with around 10-6 to 5×10-8 atmospheric pressure of air. An Al cathode plate at one end of the tube created a beam of electrons, which struck a Pt anode target or anticathode at the center generating x-rays. The anode surface was angled so that the x-rays would radiate through the side of the tube. The cathode was concave so that the electrons were focussed on a small (~1 mm) spot on the anode, approximating a point source of x-rays, which resulted in sharper images. The tubes had a third electrode, an 'auxiliary anode' connected to the first anode, but its usefulness was doubtful. To operate, a DC voltage of a few kilovolts to as much as 100 kV was applied between the anodes and the cathode, usually generated by an induction coil or sometimes an electrostatic machine. This accelerated the small number of ions present in the gas, created by natural processes like radioactivity. These struck gas atoms, knocking electrons off them, creating more positive ions in a chain reaction. All the positive ions were attracted to the cathode. When these accelerated high-speed electrons struck the atoms of the anode, they created x-rays. Crookes tubes were unreliable. As time passed, the residual air would be absorbed by the walls of the tube, reducing the pressure. This increased the voltage across the tube, generating 'harder' x-rays, until eventually the tube stopped working. To prevent this, a small device releasing a small amount of gas when heated, restoring the correct pressure.

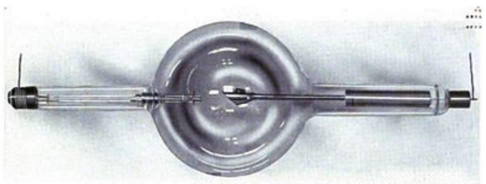
Modern x-ray tubes

The Crookes tube was improved by William Coolidge in 1913. The Coolidge tube, also called hot cathode tube, is the most widely used. It works with a very good quality vacuum (about 10-4 Pa, or 10-6 Torr, or 10-9 Atm). In the Coolidge tube, the electrons are produced by thermionic effect from a tungsten filament heated by an electric current. The filament is the cathode of the tube. The high voltage potential is between the cathode and the anode, the electrons are thus accelerated, and then hit the anode.

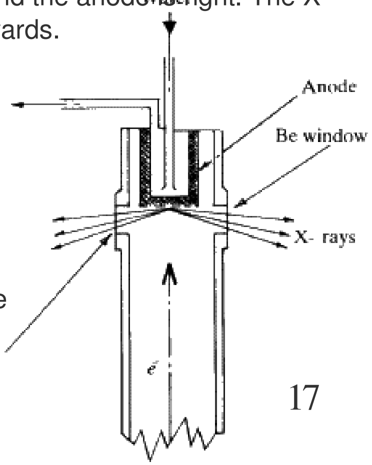
Usually in this (side-window) tubes we find an Electrostatic Lens to focus the beam onto a very small spot on the anode The anode is specially designed to dissipate the heat and wear resulting from this intense focused barrage of electrons: Mechanically spun to increase the area heated by the beam. Cooled by circulating coolant.The anode is precisely angled at 1-20 degrees off perpendicular to the electron current so as to allow escape of some of the X-ray photons which are emitted essentially perpendicular to the direction of the electron current. The anode is usually made out of tungsten or molybdenum. The tube has a window designed for escape of the generated X-ray photons. The power of a Coolidge tube usually ranges from 1 to 4 kW.



Coolidge side-window tube (scheme)
K: filament
A: anode
Win and Wout: water inlet and outlet of the cooling device (C)



Coolidge X-ray tube, from around 1917. The heated cathode is on the left, and the anode is right. The X-rays are emitted downwards.

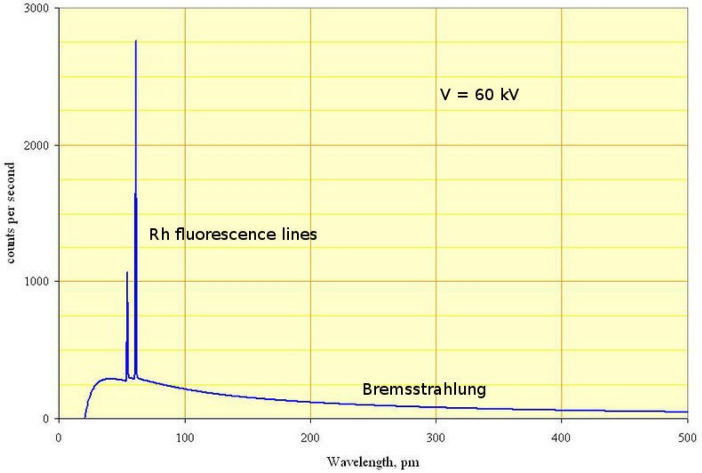


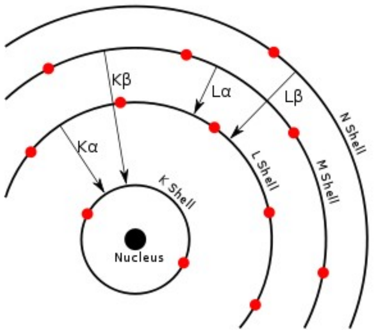
Modern x-ray tube

X-ray sources

The x-ray emission spectrum at the anode consists basically in two components: the Bremsstrahlung (deceleration) radiation and the x-ray fluorescence pattern. The first is a continuous spectrum extending up to the limiting electron energy (for example 20 keV if the potential difference between the anode and the cathode is 20 kV). The x-ray fluorescent pattern is instead composed by well-defined peaks located at energy (or wavelengths) typical of the atomic constituents of the anode. These energies reflects the electronic structure of the atoms.

The origin of the Bremsstrahlung is due to the deceleration imposed to the electrons arriving inside the anode from the surface, while the x-ray fluorescence pattern is due to the emission of photons as a result of a de-excitation of the electrons of the atomic core levels (originally excited as a result of the "collisions" with the electrons of the incoming beam.





Rh fluorescence lines:

A

Kα1	20216 eV	0.613 Ang
Kα2	20074 eV	0.618 Ang
Kβ1	22724 eV	0.546 Ang

Compton's set-up

In 1923 A.H. Compton (Nobel prize, 1927) realizes a new x-ray set-up able to measure the energy (frequency) of the photons, along with their direction.The basic set-up consists in an advanced high-power x-ray tube (target Mo, Kα λ= 0.708 Ang.), whose emission is collimated onto a graphite target.
A goniometer was used to measure the flux of the photons for different scattering angles. Moreover, a monochromator was also used to measure the wavelength dispersion of the scattered photons at a given angle. The intensity of the beam was measured using a ionization chamber. This kind of set-up provides all of the necessary information about the dynamics of the photon-graphite (electrons) interaction.

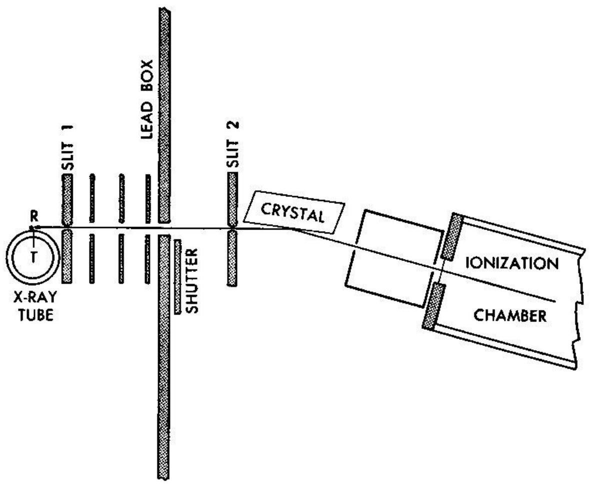
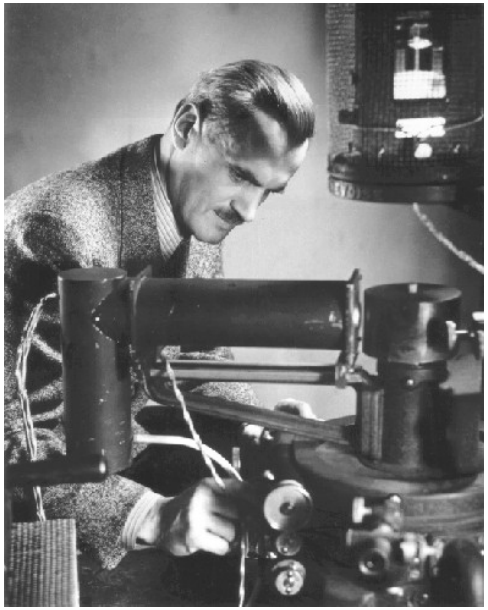


Fig. 1. Measuring the wavelength of scattered x-rays.



Compton's experiment

The main results are reported in the figures showing that while we have a constant wavelength for the incoming beam (λ_0=0.708 Ang.), we measure a clear shift for a portion of the scattered photon at longer wavelengths, increasing as a function of the scattering angle (Compton shift).

This result can not be explained by classical arguments, because the oscillating incoming field can induce a vibration of the electrons (like in a radio-antenna) of the same frequency only!

Compton explained this effect by assuming that the x-ray beam corresponds to a flux of photons of given energy and that some of those photons will interact with the free electrons (loosely bound) of the target.

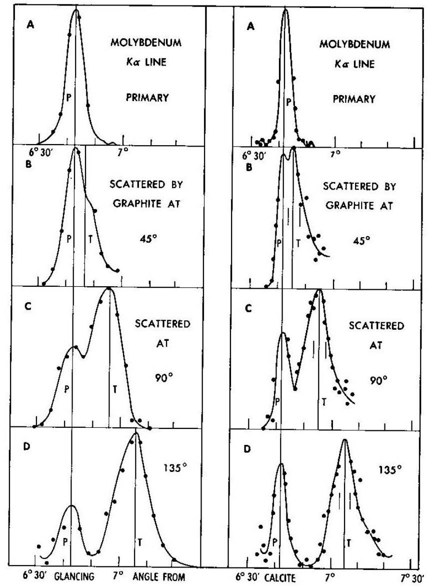
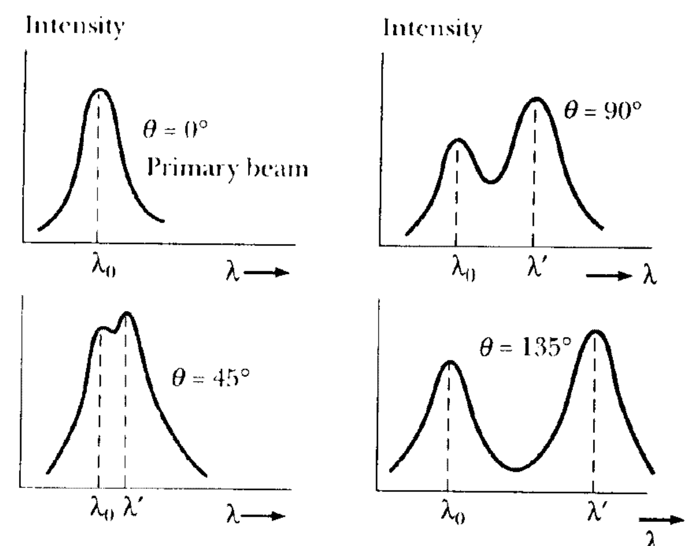


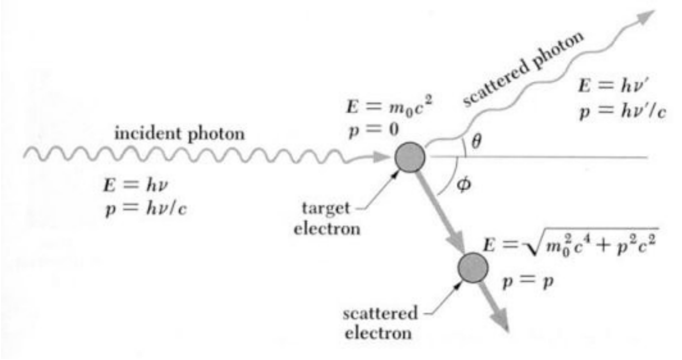
Fig. 3 Fig. 4



(b)

Collision Analysis

Quantitative analysis in terms of an inelastic photon-electron interaction can be performed applying the basic conservation principles (Energy, momentum).



Conservation of momentum gives:

$$\boldsymbol{p} = \boldsymbol{p'} + \boldsymbol{p}_e$$

where ***p*** is the momentum of the incoming photon, ***p'*** is the momentum of the scattered photon and $\boldsymbol{p}_e$ is the momentum of the scattered electron. The square gives:

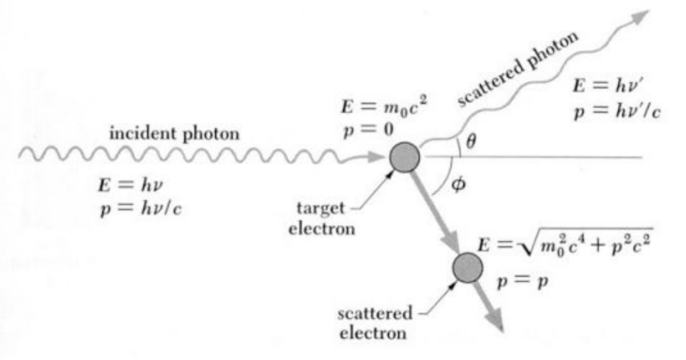
$$p_e^2 = p^2 + p'^2 - 2\,\boldsymbol{p}\cdot\boldsymbol{p'} = p^2 + p'^2 - 2\,p\,p'\cos(\theta) \quad (\mathbf{1})$$

Here the (unknown) p_e is expressed in terms of the photon momenta and of the scattering angle.

Collision Analysis

Conservation of energy gives:

$$pc + E_0 = p'c + \sqrt{E_0^2 + p_e^2 c^2}\,;\; E_0 = m_0 c^2$$



Transposing the term p'c and squaring we obtain:

$$E_0^2 + c^2(p - p')^2 + 2\mathrm{cE}_0(p - p') = E_0^2 + p_e^2 c^2$$

So we obtain for p_e:

$$p_e^2 = p^2 + p'^2 - 2pp' + 2E_0\frac{(p - p')}{c} \quad (\mathbf{2})$$

P_e can be thus eliminated using the two conservation laws (**1**) and (**2**) obtaining

$$E_0\frac{(p - p')}{c} = pp'(1 - \cos(\theta))$$

Compton shift

We wish to express our results in terms of the wavelength so we multiply each term by hc/pp'E_0, using λ = h/ p and we obtain:

$$\lambda' - \lambda = \frac{hc}{E_0}(1 - \cos\theta)$$

And the Compton shift becomes:

$$|\Delta\lambda| = \frac{h}{m_0 c}(1 - \cos\theta) = \lambda_c(1 - \cos\theta)$$

The Compton wavelength corresponds to that of the photon having the energy of the rest mass of the electron:

$$h\nu_c = \frac{hc}{\lambda_c} = m_0 c^2 = 511\, keV$$

Energy and momentum of the recoil electron are small but have been studied in a “cloud” chamber, showing that the collision is described by these equations (Compton and Simon, 1925).
Bethe and Geiger also showed) that electrons and photons were emitted simultaneously in the Compton scattering by a coincidence technique.